If $x = e^t$, write and prove a formula for $\frac{d^3y}{dx^3}$ in terms of x, $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ and/or $\frac{d^3y}{dt^3}$. **SCORE: _____ / 4 PTS**

Your proof may use the formula for $\frac{d^2y}{dx^2}$ from lecture without proving it.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[-\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}\right]$$

$$= \frac{2}{x^3}\frac{dy}{dt} \underbrace{- \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}}_{1} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \underbrace{\frac{1}{x^2}\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dx}}_{1}$$

$$= \frac{2}{x^3}\frac{dy}{dt} - \frac{1}{x^2}\frac{d^2y}{dt^2}\frac{1}{x} - \frac{2}{x^3}\frac{d^2y}{dt^2} + \frac{1}{x^2}\frac{d^3y}{dt^3}\frac{1}{x}$$

$$= \frac{1}{2}\ \underbrace{\frac{2}{x^3}\frac{dy}{dt}}\ 1\ \underbrace{- \frac{3}{x^3}\frac{d^2y}{dt^2}} + \underbrace{\frac{1}{x^3}\frac{d^3y}{dt^3}}\ \frac{1}{2}$$

Find the general solution of $3y''' + 8y'' + 12y' - 5y = 0$.

SCORE: _____ / 6 PTS

$$\underbrace{3r^3 + 8r^2 + 12r - 5 = 0}_{} \quad \frac{1}{2}$$

$$\begin{array}{c|cccc} \frac{1}{3} & 3 & 8 & 12 & -5 \\ & & 1 & 3 & 5 \\ \hline & 3 & 9 & 15 & 0 \end{array}$$

$$1 \quad \underbrace{(r - \tfrac{1}{3})}_{}\underbrace{(3r^2 + 9r + 15)}_{} = 0 \quad 1$$

$$(3r-1)(r^2 + 3r + 5) = 0$$

$$r = \frac{-3 \pm \sqrt{-11}}{2} = \underbrace{-\frac{3}{2} \pm \frac{\sqrt{11}}{2} i}_{} \quad \frac{1}{2}$$

$$y = \underbrace{Ae^{\frac{1}{3}x}}_{} + \underbrace{Be^{-\frac{3}{2}x} \cos \frac{\sqrt{11}}{2} x}_{} \quad 1$$

$$+ \underbrace{Ce^{-\frac{3}{2}x} \sin \frac{\sqrt{11}}{2} x}_{} \quad 1$$

1

Use the Wronskian to determine if the functions $f_1(x) = x^2 - 2x$, $f_2(x) = 1 - 2x^2$ and $f_3(x) = 1 - 4x$ **SCORE: _____ / 4 PTS**

are linearly independent on $(-\infty, \infty)$. **State your justification and conclusion BRIEFLY & clearly.**

$$\begin{vmatrix} x^2-2x & 1-2x^2 & 1-4x \\ 2x-2 & -4x & -4 \\ 2 & -4 & 0 \end{vmatrix} = 0 + (-8)(1-2x^2) - 4(1-4x)(2x-2)$$

$$- (-8x)(1-4x) - 16(x^2-2x) - 0$$

$$= \underbrace{-8+16x^2}_{\frac{1}{2}} \underbrace{+8-40x+32x^2}_{1} \underbrace{+8x-32x^2}_{\frac{1}{2}} \underbrace{-16x^2+32x}_{\frac{1}{2}}$$

$$\underbrace{\neq 0}_{\frac{1}{2}}$$

1 $\underline{\text{NOT LINEARLY INDEPENDENT}}$

BONUS QUESTION: Find all values of $\ln(-1)$.

SCORE: _____ / 3 PTS

$e^{ix} = \cos x + i \sin x = -1 \longrightarrow \cos x = -1 \longrightarrow x = \pi + 2n\pi$

$\sin x = 0$

$\ln(-1) = \underbrace{i\pi}_{1\frac{1}{2}} + \underbrace{2ni\pi}_{1\frac{1}{2}}$

$y_1 = e^{-x}$ is a solution of $xy'' + (2x-1)y' + (x-1)y = 0$. Find a second linearly independent solution. **SCORE: _____ / 6 PTS**

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - v'e^{-x} - v'e^{-x} + ve^{-x}$

$= v''e^{-x} - 2v'e^{-x} + ve^{-x}$

$\underline{x(v''e^{-x} - 2v'e^{-x} + ve^{-x})}$ (1)

$\underline{+(2x-1)(v'e^{-x} - ve^{-x})}$ (1)

$\underline{+(x-1)(ve^{-x})}$ ($\frac{1}{2}$)

$= e^{-x}(xv'' - 2xv' + xv + (2x-1)v' - (2x-1)v + (x-1)v)$

$= \underline{\cancel{e^{-x}}(xv'' - v')} = 0$ (1)

LET $u = v'$

$xu' - u = 0$

$xu' = u$

$\underline{\frac{1}{u}\,du = \frac{1}{x}\,dx}$ (1)

$\ln|u| = \ln|x|$

$v' = \underline{u = x}$ ($\frac{1}{2}$)

$\underline{v = \frac{1}{2}x^2}$ ($\frac{1}{2}$)

$y_2 = \underbrace{\frac{1}{2}x^2e^{-x}}_{(\frac{1}{2})}$ OR $\underbrace{x^2e^{-x}}_{(\frac{1}{2})}$

$y = x$ is a particular solution of $3x^2 y'' + 8xy' - 2y = 6x$. SCORE: _____ / 10 PTS

[a] By inspection, find a particular solution of $3x^2 y'' + 8xy' - 2y = 1$.

$-2y = 1 \longrightarrow y = -\frac{1}{2}$ $\frac{1}{2}$

[b] **Using superposition and linearity**, find the general solution of $3x^2 y'' + 8xy' - 2y = -12x + 3$.

$-2(6x)$

$3(1)$

① $3r^2 + 5r - 2 = 0$

$(3r-1)(r+2) = 0$

$r = \frac{1}{3}, -2$ ①

$y = -2(x) + 3(-\frac{1}{2}) + Ax^{\frac{1}{3}} + Bx^{-2}$

$= -2x - \frac{3}{2} + Ax^{\frac{1}{3}} + Bx^{-2}$ ① ① ①

[c] Solve the initial value problem, $3x^2 y'' + 8xy' - 2y = -12x + 3, \quad y(-1) = 1, \quad y'(-1) = -2$.

$y' = -2 + \frac{1}{3}Ax^{-\frac{2}{3}} - 2Bx^{-3}$ ①

$2 - \frac{3}{2} - A + B = 1$

$-2 + \frac{1}{3}A + 2B = -2$

$-A + B = \frac{1}{2}$

$\frac{1}{3}A + 2B = 0$

$A = -6B$

$7B = \frac{1}{2}$

$B = \frac{1}{14}$

$A = -\frac{3}{7}$

$y = -2x - \frac{3}{2} - \frac{3}{7}x^{\frac{1}{3}} + \frac{1}{14}x^{-2}$ $\frac{1}{2}$ ① ①

[d] According to the existence and uniqueness theorem,
what is the largest interval over which the initial value problem in [c] is guaranteed to have a unique solution ?

$y'' + \frac{8}{3x}y' - \frac{2}{3x^2}y = -\frac{4}{x} + \frac{1}{x^2}$

COEF'S DISCONT @ $x = 0$

① $(-\infty, 0)$